



APPLICATION OF ODD HARMONIOUS LABELLING OF GRAPHS

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ABSTRACT

The labelling of discrete structures is an attractive research topic due to its vast range of applications. The current research is looking on strange harmonious labelling. If there exists an onto $f:V(G) \rightarrow \{0,1,2,\dots,2q-1\}$ such that the induced function $f^*:E(G) \rightarrow \{1,3,\dots,2q-1\}$ defined by $f(uv) = f(u) + f(v)$ is a bijection, the graph G is said to be odd harmonious. We discuss a variety of graphs with unusually synchronized labelling. As we show, Bistar B_n , shadow n 's graph, and splitting graph are all odd harmonic graphs. We also show that the random super subdivision of path P_n allows for odd harmonious labelling. We further prove that for $n \equiv 0 \pmod{4}$ the graph $H(n,n)$ and the joint sum of two copies of cycle C_n are odd harmonic graphs.

Keywords: Harmonious labeling (HL), shadow graph, splitting graph, odd HL.

I. INTRODUCTION

The labelling of discrete structures is an attractive research topic due to its vast range of applications. The current research is looking on strange harmonious labelling. If there exists an injection $f:V(G) \rightarrow \{0,1,2,\dots,2q-1\}$ such that the induced function $f^*:E(G) \rightarrow \{1,3,\dots,2q-1\}$ defined by $f(uv) = f(u)+f(v)$ is a bijection, the graph G is said to be odd harmonious.

A few graphs with remarkably harmonious labelling are examined. We demonstrate that b is the graph of $B(n,n)$ shadow) and that splitting graphs are odd harmonic graphs. We also show that the arbitrary super subdivision of path P_n allows for odd harmonious labelling. We also prove that the graph $H(n,n)$ is an odd harmonic graph, as is the joint sum of two copies of cycle C_n for $n \equiv 0 \pmod{4}$. $G = (V(G),E(G))$ is a simple, finite, connected, and undirected graph with $|V(G)| = p$ and $|E(G)| = q$.

For terminology and notation, Gross and Yellen [5] are the gold standard. For the present investigations, we will provide a brief review of definitions and other essential facts.

We show that the number of edges in any odd harmonic Eulerian graph is equivalent to either 0 or 2 (mod 4) and that the converse is incorrect. Odd harmonic graphs generated by two copies of the even cycle C_n sharing a shared edge were also illustrated. Furthermore, for graphs produced by two copies of cycle C_n sharing a same vertex, we discovered an odd harmonic labelling where n is equal to $N \equiv 0 \pmod{4}$.

We also show that the Cartesian combination of cycle graph C_m and route P_n is an odd harmonic graph for each $n \geq 2, m \equiv 0 \pmod{4}$. Finally, we present a large number of novel families of odd harmonious graphs.

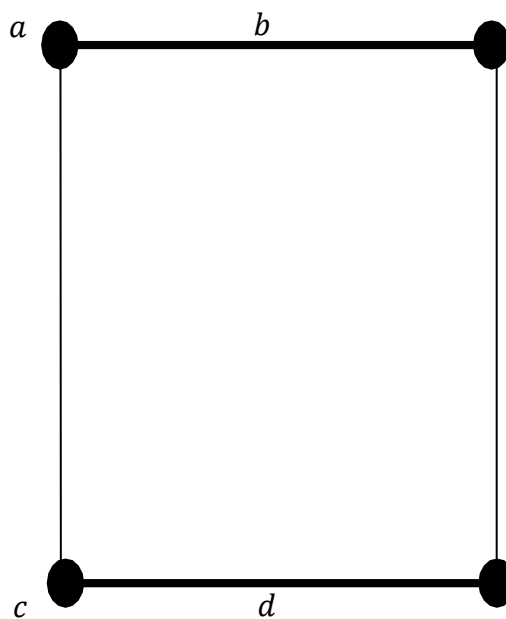
II. BASIC DEFINITIONS

The following are some basic definitions that were employed in this investigation.

Definition– 2.1:

A graph $G = (V, E)$ has a set V of vertices (also known as nodes) and an edge set E .

Example:



Fig–2.1

The edges of the graph in the example above are ab , ac , cd , and bd . The graph's vertices are a , b , c , and d , respectively.

Definition– 2.2:

A bipartite graph is one in which every edge of E connects a vertex in v_1 to a vertex in v_2 in a simple graph $G = (V, E)$ with vertex partition $V = v_1, v_2$.

A bipartite graph, in general, comprises two sets of vertices, let's call them v_1 and v_2 , and an edge should connect any vertex in set v_1 to any vertex in set v_2 .

Example:

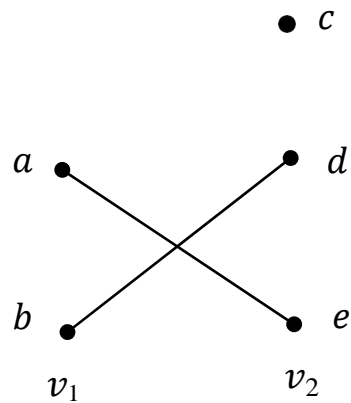


Fig-2.15

Definition- 2.2:

A bipartite graph 'G', $G = V, E$ with partition $V = \{v_1, v_2\}$ is said to be a **complete bipartite graph** if every vertex in v_1 is connected to every vertex of v_2 . A full bipartite graph, in general, connects every vertex from set v_1 to every vertex from set v_2 .

The following graph is a complete bipartite graph because it has edges connecting each vertex from set v_1 to each vertex from set v_2 .

Example:

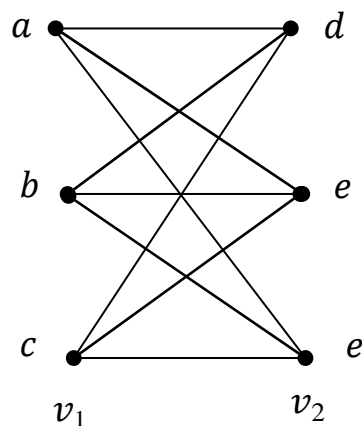


Fig- 2.16

Definition– 2.3:

Graceful labelling of a graph G is a function f if $f: V(G) \rightarrow \{0, 1, 2, \dots, q\}$ is injective, and the function that is induced $f^*: E(G) \rightarrow \{1, 2, \dots, q\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is one to one & onto.

Definition– 2.4:

If a graph G has a one-to-one function, it is said to be harmonic if $f: V(G) \rightarrow Z_q$ such that $f^*: E(G) \rightarrow Z_q$ defined by $f^*(uv) = (f(u) + f(v)) \pmod{q}$ is a one to one & onto and f is said to be *harmonious labeling* of G .

Definition– 2.5:

Let k, R be integers with $R \geq 1$ and $k \leq R$. A graph G is said to be *Smarandachely R-harmonious labeling* if there exist an injection $f: V(G) \rightarrow \{0, 1, 2, \dots, kq-1\}$ as a result of which the induced function $f^*: E(G) \rightarrow \{1, R+1, \dots, pq-1\}$ defined by $f^*(uv) = f(u) + f(v)$ is a one to one & onto.

Definition– 2.6:

If $p = k = 2$, such a 2-harmonious labelling of G , f , and g is known as an odd harmonious labelling of G , f , and g . Those functions are known as the vertex and edge functions, respectively.

Definition– 2.7:

A linked graph G 's shadow graph $D_2(G)$ is created by making two copies of G , G' and G'' . Join the neighbours of each vertex u' in G' to the corresponding vertex v' in G'' .

Definition– 2.8:

The splitting graph $S'(G)$ of a graph G is obtained by adding a new vertex to the graph G .

$N(v) = N(v')$, with v' corresponding to each vertex v in G .

III. ODD HARMONIOUS LABELING OF SOME GRAPHS

Theorem 2.1:

$D_2(B_{n,n})$ is an odd harmonious graph.

Proof:

Consider two copies of $B_{n,n}$.

Let $\{u, v, u_i, v_i, 1 \leq i \leq n\}$ and $\{u', v', u'_i, v'_i, 1 \leq i \leq n\}$ be the corresponding vertex sets of each copy of $B_{n,n}$. Denote $D_2 B_{n,n}$ as G . Then

$$|V(G)| = 4(n+ 1) \text{ and } |E(G)| = 4(2n + 1).$$

Consider $f : V(G) \rightarrow \{0, 1, 2, 3, \dots, 16n+7\}$,

We consider following two cases.

Case1: n is even

$$f(u_i) = 9 + 4(i - 1), 1 \leq i \leq n, f(u'_i) = f(u_n) + 4i, 1 \leq i \leq n,$$

$$f(v_1) = f(u'_n) + 3, f(v_{2i+1}) = f(v_1) + 8i, 1 \leq i \leq \frac{n-1}{2},$$

$$f(v_2) = f(u'_n) + 5, f(v_{2i}) = f(v_2) + 8(i - 1), 2 \leq i \leq \frac{n}{2},$$

$$f(v'_1) = f(v_n) + 6, f(v'_{2i+1}) = f(v'_1) + 8i, 1 \leq i \leq \frac{n-1}{2},$$

$$f(v'_2) = f(v_n) + 8, f(v'_{2i}) = f(v'_2) + 8(i - 1), 2 \leq i \leq \frac{n}{2}$$

Case2: n is odd

$$f(u_i) = 9 + 4(i - 1), 1 \leq i \leq n, f(u'_i) = f(u_n) + 4i, 1 \leq i \leq n,$$

$$f(v_1) = f(u'_n) + 3, f(v_{2i+1}) = f(v_1) + 8i, 1 \leq i \leq \frac{n-1}{2},$$

$$f(v_2) = f(u'_n) + 5, f(v_{2i}) = f(v_2) + 8(i - 1), 2 \leq i \leq \frac{n-1}{2},$$

$$f(v'_1) = f(v_n) + 2, f(v'_{2i+1}) = f(v'_1) + 8i, 1 \leq i \leq \frac{n-1}{2},$$

$$f(v'_2) = f(v_n) + 8, f(v'_{2i}) = f(v'_2) + 8(i - 1), 2 \leq i \leq \frac{n-1}{2},$$

The vertex function f defined above induces a bijective edge function $f^*:E(G)\rightarrow\{1,3,\dots,16n+7\}$. As a result, f is an unusually harmonic labelling for $G = D_2(B_{n,n})$.

Theorem3.2:

$S'(B_{n,n})$ is an odd harmonious graph.

Proof:

Consider $B_{n,n}$ with vertex set $\{u, v, u_i, v_i, 1 \leq i \leq n\}$, where u_i, v_i are pendant vertices. In order to obtain $S'(B_{n,n})$, and u', v', u'_i, v'_i vertices corresponding to u, v, u_i, v_i where, $1 \leq i \leq n$. If $G = S'(B_{n,n})$

Then $|V(G)| = 4(n+1)$ and $|E(G)| = 6n+3$.

Define vertex labeling $f: V(G) \rightarrow \{0, 1, 2, 3, \dots, 12n+5\}$ as follows.

$$\begin{aligned}
 f(u_i) &= 7 + 4(i - 1), 1 \leq i \leq n, f(v_1) = f(u_n) + 3, f(v_{i+1}) \\
 &= f(v_1) + 4i, 1 \leq i \leq n-1, \\
 f(u'_1) &= f(v_n) + 5, f(u'_{i+1}) = f(u'_1) + 2i, 1 \leq i \leq n-1, \\
 f(v'_1) &= f(u'_n) - 1, f(v'_{i+1}) = f(v'_1) + 2i, 1 \leq i \leq n-1.
 \end{aligned}$$

The vertex function f defined above induces bijective edge function $f^*:E(G)\rightarrow\{1,3,\dots,12n+5\}$. As a result, f is an unusually harmonious labelling for $G = S(B_{n,n})$.

Theorem3.3:

Arbitrary super sub division of path P_n is an odd harmonious graph.

Proof:

Let P_n represent the path with n vertices and v_i ($1 \leq i \leq n$) represent the vertices of P_n .

Arbitrary super subdivision of P_n is obtained by replacing every edge e_i of P_n with K_{2,m_i} and we denote this graph by G .

Let u_{ij} be the vertices of m_i -vertices part of K_{2,m_i} where $1 \leq i \leq n-1$ and $1 \leq j \leq$

$$\max\{m_i\}_{i=1}^{n-1} \text{ let } \alpha = \sum^{n-1} m_i \text{ and } q = 2\alpha.$$

We define vertex labeling $f: V(G) \rightarrow \{0, 1, 2, 3, \dots, 2q-1\}$ as follows.

$$f(v_{i+1}) = 2i, 1 \leq i \leq n-1,$$

$$f(u_{1j}) = 1 + 4(j-1), 1 \leq j \leq m_1,$$

$$f(u_{ij}) = f(u_{(i-1)n}) + 2 + 4(j-1), 1 \leq j \leq m_i, 2 \leq i \leq n.$$

The above-mentioned vertex function f generates a bijective edge function $f^*: E(G) \rightarrow \{1, 3, \dots, 2q-1\}$. As a result, f is a strangely harmonic labelling of G . As a result, path P_n 's arbitrary super subdivision produces an oddly harmonic graph.

Theorem 3.4:

The addition of two copies of C_n yields an unusually harmonious labelling form $\equiv 0 \pmod{4}$.

Proof:

The vertices of the first copy of C_n are denoted by v_1, v_2, \dots, v_n , while the vertices of the second copy are denoted by $v_{n+1}, v_{n+2}, \dots, v_{2n}$. Connect the two copies of C_n with a new edge, and call the resulting graph G then $|V(G)| = 2n$ and $|E(G)| = 2n+1$.

Without loss of generality we assume that the new edge by $v_n v_{n+1}$ and

$v_1, v_2, \dots, v_n, v_{n+1}, v_{n+2}, \dots, v_{2n}$ will form a spanning path in G .

Define $f: V(G) \rightarrow \{0, 1, 2, 3, \dots, 4n+1\}$ as follows.

$$f(v_{2i+1}) = 2i, 0 \leq i \leq \frac{3n}{4} - 1,$$

$$f(v_{\frac{3n}{2} + 2i - 1}) = \frac{3n}{2} + 2i, 0 \leq i \leq \frac{n}{4},$$

$$f(v_{2i}) = 1, \text{ for } 1 \leq i \leq \frac{n}{4},$$

$$f(v_{\frac{n}{2} + 2i + 1}) = \frac{n}{2} + 3 + 2i, 0 \leq i \leq \frac{3n}{4} - 1.$$

The above-mentioned vertex function f generates a bijective edge function $f^*: E(G) \rightarrow \{1, 3, \dots, 4n+1\}$. As a result, f is a strangely harmonic labelling of G . As a result, the addition of two copies of C_n yields an oddly harmonic labelling for $n \equiv 0 \pmod{4}$.

Theorem3.5:

The graph $H_{n,n}$ is an odd harmonious graph.

Proof:

Let $V = \{v_1, v_2, \dots, v_n\}$, $U = \{u_1, u_2, \dots, u_n\}$ be the partition of $V(H_{n,n})$.

Let $G = H_{n,n}$ then $|V(G)| = 2n$ and $|E(G)| = \frac{n(n+1)}{2}$.

Define odd harmonious labeling $f: V(G) \rightarrow \{0, 1, 2, 3, \dots, (n^2+n-1)\}$ as follows.

$$f(v_i) = i(i-1) \text{ for } 1 \leq i \leq n,$$

$$f(u_i) = (2n+1), \text{ for } 1 \leq i \leq n$$

The above-mentioned vertex function f generates a bijective edge function $f^*: E(G) \rightarrow \{1, 3, \dots, n^2+n-1\}$. G is labelled with an oddly harmonious labelling of f .

Theorem3.6:

If G is an odd Eulerian graph with q edges that is harmonic, then $q \equiv 0 \text{ or } 2 \pmod{4}$.

Proof:

Let G be an odd harmonious graph, and let $f: V(G) \rightarrow \{0, 1, 2, \dots, 2q-1\}$ be an odd harmonious labeling for G .

Since G is a graph then $\sum (f(v_i) + f(v_j)) = 2k$, k is a constant.

For each $v_i, v_j \in V(G)$, $\sum (f(v_i) + f(v_j)) = 2k$, So $1+3+5+\dots+2q-1 = 2k$, this implies that

$$\frac{q(1+2q-1)}{2} = 2k.$$

Therefore $q \equiv 0 \text{ or } 2 \pmod{4}$.

Theorem3.7:

Even cycle is duplicated twice. When C_n share a similar vertex, the graph becomes strangely harmonious $n \equiv 0 \pmod{4}$.

Proof:

Let v_1, v_2, \dots, v_n be the cycle's vertices $C_n, n \equiv 0 \pmod{4}$.

Consider the graph G , which is made up of two copies of C_n that share a vertex $|V(G)| = 2n - 1$

And $|E(G)| = 2n - 3$,

Let's call this vertex v_1 to keep things simple. Allow G to be described as shown in Figure.

(3.6).

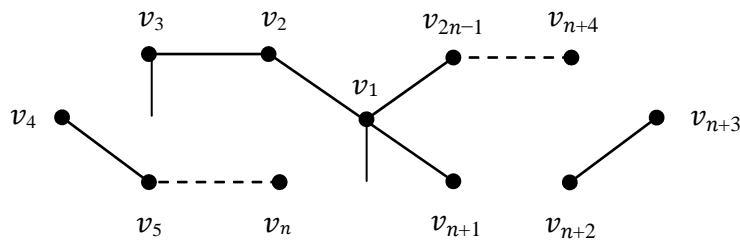


Fig-3.6

The labelling function is defined.

$$f:V(G) \rightarrow \{0,1,2,3,\dots,4n-1\}$$

As follows:

for $i \in [1, n + 1]$:

$$f(v_i) = (i-1),$$

$$i+1, \text{ (ieven),}$$

For $i = n+1$:

For $n+2 \leq i \leq 2n-1$:

$$f(v_i) = f(v_{n+1}) = 2n+3,$$

$$i-2, \text{ when } i = n+2, n+4, \dots, \frac{3n}{2}, 2$$

$$i, \text{ when } i = \frac{3n}{2} + 2, \frac{3n}{2} + 4, \dots, 2n-1,$$

$$i-2, \text{ when } i = n+3, n+5, \dots, 2n-1.$$

IV. CONCLUSION

P_n , $B(n,n)$, and C_n are odd harmonic graphs for all n , with C_n being an odd harmonious graph for $n \equiv 0 \pmod{4}$, and $B(n,n)$ shadow and Γ 's splitting graphs allowing odd harmonious labelling. As a result, odd harmony stays invariant for B 's shadow graph and splitting graph. When P_n is freely subdivided, it remains invariant. A future research area could be comparing findings for different graph families and in the context of diverse graph labelling challenges. Because labelled graphs are used in so many diverse applications, such as communications networks, circuit design, coding theory, radar, astronomy, X-ray, and crystallography, it is better to have generalised discoveries or results for an entire class. Several families of odd harmonious graphs are discussed in this paper. Comparable results for other graph families, as well as in the context of different labelling procedures, are still a work in progress.

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